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RESTITUTION

OF THE

GEOMETRICAL TREATISE OF APOLLONIUS PERGÆUS

ON

INCLINATIONS.

ALSO THE

THEORY OF GUNNERY;

OR THE

DOCTRINE OF PROJECTILES

IN A NON-RESISTING MEDIUM.

BY REUBEN BURROW.

L O N D O N :

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M,DCC,LXXIX.



TO THE
RIGHT HONOURABLE
LORD VISCOUNT TOWNSHEND,
MASTER GENERAL OF THE ORDNANCE,
&c. &c. &c.

THE FOLLOWING PIECES ARE DEDICATED BY

HIS LORDSHIP'S MOST OBEDIENT

AND MOST OBLIGED HUMBLE SERVANT,

REUBEN BURROW.



P R E F A C E.

AS the famous Problem of the Ancients concerning Inclinations (or the method of drawing a right line through a given point, so that the part thereof intercepted between two given lines may be of an assigned length), does not appear to have been publicly considered, since the days of Apollonius, by any other authors than Ghetaldus and Dr. Horfley; and as it is allowed, by the best judges among the Moderns, to be of the greatest utility in geometrical constructions, and was honoured by the Ancients with a distinguished rank among their elementary treatises of Composition and Resolution; it is presumed that the following attempt to render the same more general and explicit, can stand in need of no other apology for its publication, than what the imperfections of its execution may require, and which the Author flatters himself are neither numerous nor of much consequence; for though the whole was begun and finished in less than a week, it was done with some degree of care: and the Author having since had a sight of Pappus's Collections, finds reason to conjecture that he has come nearer to the spirit of the great original, than the production of the Reverend Compiler. For as to the work of Dr. Horfley, it is split into such an infinity of different cases, frittered into so many divisions and sub-divisions, and treated besides in a manner so closely bordering upon Algebra, that it does not appear to have the least similarity to any of the genuine productions of Apollonius; and what makes it still more defective, he has not only

left his Constructions undemonstrated, but has entirely omitted those very material propositions which make the third and fourth Problems of the following book; not to mention the inelegance of his method, his virulent remarks, and his arrogant and contemptuous expressions against former writers. The Author therefore hopes, that if the following should meet with the approbation of Mathematicians, any apology for treating the same subject after the Doctor will be entirely needless.

A N

A N

E S S A Y, &c.

L E M M A.

1. **I**F DC and Qf are two right lines meeting in A; and if CA and AD differ less from equality than QA and Af; then if the rectangle DAC be equal to the rectangle fAQ, DC will be less than Qf. See Fig. I.

Let CD and Qf be bisected in m and n; then because CA differs less from AD than QA from Af, Am is less than An; but $CAD + Am^2 = mD^2$ and $QAf + An^2 = nf^2$; therefore Dm is less than nf, and consequently DC is less than Qf.

COROLLARY. Hence if AQ be greater than AC, AD will be greater than Af, and the contrary.

2. IF the rectangle DAC be equal to QAf, and AD be less than Af; DC will be greater than Qf, and the nearer AC and AD approach to equality, the less will CD become. See Fig. II.

For if DC be not greater than Qf, it must either be equal or less, and because AD is less than Af, the sum AC is less than AQ; and therefore the rectangle CAD is less than QAf, which is absurd: the other part is self-evident.

In the same manner, if the rectangle DAC be equal to QAf, and AC be greater than AQ, Af will be greater than AD, and CD greater

greater than Qf ; also when AD is the least, DC will be the greatest; and when CD is greater than Qf , AC will be greater than AQ ; and *vice versa*.

P R O B L E M I.

IF FB be a given circle, and CQ a line at right angles to its diameter; required to draw AQ through the intersection of the diameter and the circumference A , so that QF may be of a given length.

ANALYSIS. Join FB , then because AFB and ACQ are right angles, and the angle A common, the rectangle $FAQ = BAC$; but FQ and the rectangle CAB are given, therefore AF is also given: Hence this CONSTRUCTION:

Let qf be the given line, and find the point a so that the rectangle $qaf = BAC$ and let AF be taken $= af$ meeting the line CQ in Q , then QF will be equal to the given line.

DEMONSTRATION. Join BF , then because AFB and ACQ are right angles, and the angle at A common, the points $BFCQ$ are in a circle, therefore the rectangle $BAC = QAF = qaf$ by construction, and $AF = af$ by construction, therefore $QF = qf$.

LIMITATIONS. *First*; Let AC be greater than AB (see *Fig. III.*), then as AF is less than AB , and $AB \cdot AC = AF \cdot AQ$, therefore by the Lemma, QF will be greater than BC , and therefore BC is its limit. *Secondly*, Let AC be less than AB , then as AF is less than AB , QF will be less than BC , and as AF decreases, FQ also decreases till they become equal, and after increases indefinitely.

In *Figure IV.* AB is either greater, equal, or less than AC : If greater, take $AN = AC$, and draw the perpendicular NF meeting the

the circle in F, and draw AFQ which will be the least line possible by the Lemma, because the rectangle BAC and QAF are equal: if AB is equal to AC, BC will be the least by the same reason; and if AB be less than AC, BC will still be the least, because while AF decreases QF increases.

SCHOLIUM. The construction of the above is similar to Mr. Simpson's, and that used in the following Problem, is not materially different from the method of Ghetaldus.

P R O B L E M II.

IF cbf and dbq are two lines given in position, meeting in b , and a a point given in the line bisecting the angle cbq ; it is required to draw a line through the given point a , meeting bc in f , and bd in q , so that qf may be of a given length. See Fig. VI. & VIII.

ANALYSIS. Suppose it done, and FBQ to be equal and similar to fbq ; then as the angle FBQ is given, the point B will fall in the circumference of a circular segment described on FQ containing the given angle FBQ; also AB meets the circle, for A is within the circle in Fig. V. and in Fig. VII. if BG was to touch the circle, FBG would be equal to FQD by Euclid, Prop. XXXII. b. iii. and as FBG is equal to CBA or ABQ, ABQ would be equal FQD, which is absurd; therefore let AB meet the circle in G and join G, Q and F, then because the angles CBA and ABQ are equal, GBF and DBG are also equal; but GBF is equal to GQF, and DBG equal to QFG, therefore GQF is equal to GFQ, and consequently the point G bisects the arch FGQ; and the line AB is also given: hence this

CONSTRUCTION. On the given line FQ describe a segment of a circle FBQ containing the given angle cbd , and bisect the arc
C FGQ

FGQ in G, then through the point G draw a line GBA; so that AB, the part intercepted, between FQ, and the circumference may be equal to ab , and join FB and BQ; and in bc take bf equal to BF, and join af meeting bd in q , then qf is equal to QF, the line required.

DEMONSTRATION. For because the arc $GQ=GF$, the angle $DBG=GFQ=GBF$, and DBC is equal dbc ; therefore ABF is equal to abf , and because AB and BF are equal to ab and bf , by construction; therefore af is equal to AF, and the angle baf equal BAF and $BAQ=baq$, consequently as ab , baq and abq are equal to AB, BAQ and ABQ, aq is equal to AQ, and therefore qf is also equal to QF.

LIMITATION. It is evident that qf is unlimited in the sixth Figure, because the sides bq , bf increase, while the angle qbf remains constant; but in Fig. VIII. draw nm through the point a perpendicular to ba , and nm will be the limit of fq ; for the angle bma is equal to bna , but bma is greater than bfa , therefore anq is greater than afm ; wherefore if the angle ans be made equal to afm the points m, f, s, n will be in a circle, and the rectangle $man=fas$; and as ma is equal to an , fs is greater than mn by the Lemma, much more fq .

P R O B L E M III.

HAVING two circles BQC and DFN, whose centres are R and S, given in magnitude and position, and also the point A, so posited in the line joining the centres, that $AB:AC::AD:AN$; it is required to draw a line through the point A meeting the circle BQC in Q and q , and the circle DFN in F and f , so that the intercepted part QF, qf , qF or Qf may be of a given length. See Fig. IX. X. XI. XII. and XIII. N. B.

N. B. When the circles are described as above, then $AC:AN :: AB:AD :: AQ:AF :: Aq:Af$, and the parts QC and FN , Bq and Df , &c. are similar parts; but QC and Df , Bq and NF , &c. are called dissimilar parts: also the rectangle $BA \cdot AN = AC \cdot AD = Aq \cdot AF = AQ \cdot Af$, &c. this follows from the Vth Prop. of Apollonius on plane places.

ANALYSIS. Suppose it done, and QF the given line; then because $AQ:AF :: AR:AS$, RQ is parallel to SF ; and therefore if RM be parallel to QF , RQ will be equal to FM ; but RQ is constant, therefore FM is constant; and SF is constant, therefore SM is constant, and consequently the locus of the points M is a circle; and the analysis is the same for qf : also $AQ \cdot Af$ is equal to $AC \cdot AD$ or $AB \cdot AN$; hence this CONSTRUCTION:

1. To have the line QF intercepted between the similar parts of a given length. Take NG from the end of the diameter of the greatest circle, equal to the radius of the other, and with the distance SG describe a circle, and from R take RM of the given length; join SM meeting the circle whose diameter is DN in F , and join AF meeting the other circle in Q , then FQ is the line required; and the process is exactly similar for qf as appears from the *Figure*.

2. To have the given line Qf or qF intercepted between the dissimilar parts. Find a point A in the given line Qf so that $QA \cdot Af$ may be equal to $AB \cdot AN$, then AQ being thus found, and the point A given, the position of AQ will be given.

DEMONSTRATION of the first part of the Construction. $AS:AR :: AF:AQ$ by Proposition V. of Apollonius, therefore by Prop. II. vi. Eucl. SF is parallel to RQ ; and $MF = RQ = NG$ by construct. therefore QF is equal and parallel to RM ; and by parity of reason, qf is equal and parallel to Rm .

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THE second part of the Construction is self-evident from the
aforesaid proposition, for $AQ \cdot Af$ is given and Qf is given by
hypothesis, therefore AQ is given.

LIMITATIONS of the first, or *similar* parts.

QF must not be greater than CN : for $RC=GN$, therefore
 $RG=CN$, but RM cannot be greater than RG , therefore QF
cannot be greater than CN .

Also qf cannot be less than BD : for $BR=Dn$, therefore Rn
 $=DB$; but Rm cannot be less than Rn , therefore Rm or qf can-
not be less than BD ; and the other limit is Tt .

LIMITATIONS of the *diffimilar* parts.

HERE AD is either less than AC or not less.

1. LET AD be less than AC , and through A draw QA meet-
ing the circle DN in f , so that $QA=Af$, then will Qf be the least
line that can be drawn through A : for $DA \cdot AC=QA \cdot Af$, and
 $Af=AQ$, therefore DC is greater than Qf by the Lemma; there-
fore Qf cannot be greater than DC when drawn on the side of Q
next C , nor greater than Tt when drawn on the other side of Q :
here Tt are the points of contact.

By parity of reason qF cannot be greater than BN , nor less than
 Tt ; for $qA \cdot AF=BA \cdot AN$ and Aq is greater than AB , there-
fore qA and AF differ less from equality than BA and AN ,
therefore qF is less than BN ; and as Aq is greatest at t , tT will
be the least by the Lemma.

2. LET AD be not less than AC ; then as AD must either be
equal to AC or greater; and as $Af \cdot AQ=AC \cdot AD$ and Af is
greater than AD ; therefore by the Lemma, Qf must be greater
than DC ; and as AQ decreases till it comes to t , so Qf increases
till it becomes tT ; therefore Qf cannot be greater than tT nor
less than CD .

ALSO

Also Fq cannot be greater than BN nor less than Tt ; for $Aq \cdot AF = BA \cdot AN$, and Aq is greater than AB , therefore qA and AF differ less from equality than BA and AN ; therefore qF is less than BN ; and as Aq increases till it is greatest at t , Tt will then be least, and consequently qF cannot be less than Tt nor greater than BN .

SCHOLIUM. It is evident that when the point A falls within the circles, fQ and qF may interchange positions; however, the limits will still be included in one or other of the cases: it is also evident that all those cases where the circles touch one another and A is the point of contact, are only particular cases of the above Problem.

P R O B L E M IV.

HAVING two circles, RBC and KFT given in magnitude and position, and also the position of a right line; it is required to draw a line parallel to this line, so that the part intercepted between the circumferences shall be of a given length.

ANALYSIS. Suppose BC parallel to the line whose position is given, and QF to be the given line, QR joined, and FD parallel thereto; then as QF is parallel to RD , and DF parallel to RQ , RD will be equal to QF and DF equal to RQ ; hence this CONSTRUCTION:

TAKE RD equal to the given line, and with the centre D and distance RQ , describe a circle cutting the circle KTF in F , and draw FQ parallel to RD : the construction of all the other cases is the same and the Demonstration is self-evident.

DETERMINATION of the Limits.

ANALYSIS. Suppose (*in Fig. XIV.*) the circle KTF to move parallel
D parallel

parallel to itself, and the line CB , along KS till it touch the other circle in Q ; and let S be its centre at that instant; then it is evident that each part of the circle has moved equal distances, and F being the point that first touches the other circle, FQ will be the least possible, and RS , the distance of the centres, is equal to the sum of the radii, also if the motion be continued till KTF touch RBC outwardly on the contrary side, as at q , it is evident that fq will be the greatest line that can be intercepted. Also in *Figure XV.* suppose the circle KTF to move parallel as before, till it touch the other circle inwardly at Q , then FQ will be the least possible; and in that case the distance of the centres RS is equal to the difference of the radii: also, if the motion be continued till the circles touch outwardly as at H , then FH will be the greatest line that can be intercepted on that side; and by the same method of reasoning, fk and fq will be the least and greatest lines that can be drawn on the other. Hence this CONSTRUCTION.

DRAW RC through the centre of one circle parallel to the line given in position, and from the other centre K with the distance KD , equal to the sum or difference of the radii, intersect DR in d and D , join K , d and D meeting the circle KTF in f and F , and through these points draw lines parallel to DR , meeting the circle RPN in Q and q , and the circle KFT in F and f , then will FQ and fk be the least lines, and fq , FH the greatest that can be intercepted between the two circles.

DEMONSTRATION. For, draw any other line mn ; then $FQ=ma$, but ma is less than mn , therefore FQ is less than mn : also fq is greater than rs , for $fq=rh$, but rh is greater than rs , therefore fq is greater than rs .

Hence also it appears that if tp and TP be tangents parallel to RD , then will tp and TP be the limits of the lines FQ and FH ;
and

and tG and TN those of fk and fq ; and the same method will serve however the figures and cases may be varied.

SCHOLIUM. This Proposition is that particular case of the general Problem in which the given point is supposed to be removed to an infinite distance.

P R O B L E M V.

HAVING two concentric circles DAS and EBQ given in position and magnitude, and also a given point P , it is required to draw a line through P , so that the intercepted part QS may be of a given length.

CONSTRUCTION. Join PC , and parallel thereto, take DE or De of the given length by the last Proposition; on De let fall the perpendicular CW , and with this distance and the centre C describe a circle, and from P draw the tangent to it PR , meeting the concentric circles in Q , q and S then will QS or qS be of the given length.

FOR $SM=DK$ and $Qq=Ee$, therefore $SR=DW$ and $RQ=WE$, and consequently $SQ=DE$ and $Sq=De$.

LIMITATION. SQ cannot be less than AB , nor greater than NT , and Sq cannot be greater than Ab , nor less than NT .

SCHOLIUM. The method is exactly the same whatever position P may take, and it is evident that when the two circles coincide, De becomes DK , and the Construction is then that particular case which makes the XLIVth and XLVth propositions in p. 192 of Gregory St. Vincent, which Dr. Horsley has, with his usual acknowledgment, made the first and second propositions of his Refutation.

PROBLEM

P R O B L E M VI.

HAVING two given circles AFB and BQD which touch each other at B, and also a given point A, made by the intersection of the line joining their centres and the circumference; it is required to draw a line through the given point A, meeting the circle whose diameter is AB in F, and the other circle in Q, so that FQ may be of a given length. See *Figures XVII. XVIII. XIX. XX. XXI. and XXII.*

ANALYSIS. Suppose that AQ, the line whose position is required, meets the circle AFB in F, and the circle BQD in P and Q, and that FQ is of the given length; bisect DB in m , and suppose BF and DP joined, and mn and DS perpendicular to AQ, also let BK be parallel to DP; then because BF, mn and SD are parallel, and $Bm = mD$, Fn is also equal to nS , but $Qn = nP$, therefore PS is equal to QF; also because $AD : AB :: AS : AF :: AP : AK$, therefore $AD : AB :: PS : FK$, but PS is equal to QF, therefore $AD : AB :: QF : FK$ and QF is given by hypothesis, therefore FK and QK are also given; and because the angle AQB is equal to ADP which is equal to ABK, therefore the triangles ABK and AQB are similar, and therefore $AQ \cdot AK = AB^2$: Hence this

CONSTRUCTION. Let qf be equal to the given line, and take fk a fourth proportional to AD, AB and qf , then find the point a so that the rectangle qak may be equal to AB^2 , and make $AQ = aq$, then QF will be the line required.

DEMONSTRATION. For join BQ and DP, and draw BK parallel to DP; then because the angle $ABK = ADP$, and $ADP = AQB$, by iii. Euclid, Prop. XXXII. therefore the triangles, ABK AQB, are similar, and therefore $AK \cdot AQ = AB^2 = ak \cdot aq$ by construction, t $AQ = aq$, therefore $AK = ak$ and $KQ = kq$; but because $Bm = mD$,

mD , Fn is equal to nS and FQ to PS ; and as the triangles KBF and PSD are similar, $AD:AB::PS:FK::QF:FK$; but by Construction $AD:AB::qf:fk$, therefore $QF:FK::qf:fk$, and consequently $QF:QK::qf:qk$, but $QK=qk$, therefore $QF=qf$.

METHOD of determining the Limitations.

ANALYSIS. Because $QF:FK::AD:AB$, therefore the ratio of QF to QK is constant, and consequently when QK is the least or greatest possible, QF will be the least or greatest also; but because $AK \cdot AQ = AB^2$ and the locus of all the points Q is a circle, therefore the locus of all the points K will be a circle similarly described with those in the first Problem, by Prop. IX of Apollonius; and therefore if the point G be so taken, that $AD:AB::AB:AG$, and upon GB a circle be described, and the limits for the circles DQB and BKG be found as in that Problem, the limits for the circles DQB and AFB will also be determined by the same operation.

HENCE FQ in *Fig. XVII.* and *XIX.* cannot be greater than the part of the tangent At to the circle BtD intercepted between the circumference BF and the point of contact t ; nor less in *Figures XVIII.* and *XX.*; for $AK \cdot AQ$ is constant in all the cases, and while AQ in *Fig. XVII.* increases, QK , by the Lemma, also increases, and consequently QF which is in a constant ratio to it; but AQ is greatest when it becomes At , therefore QF is then greatest also; and the contrary takes place in *Fig. XVIII.*; also in *Fig. XIX.* QK (by the Lemma) increases while AQ decreases, and when AQ is least, or becomes At , QK and consequently QF will then be greatest; and in *Fig. XX.* the reverse takes place, for there AQ is least at AD and greatest at At , and therefore the limits of QF are DB and tH , supposing the tangent At to meet the circle AHB in H : also in *Fig. XXI.* QK has a constant ratio to

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Q F

QF and $AQ \cdot AK = AB^2$; therefore while AQ decreases, QK and consequently QF increases; but AQ is least at AH, therefore QK and consequently QF is then greatest, and by the same reason, in Fig. XXII. QF cannot be less than AH, nor greater than DB.

P R O B L E M VII.

HAVING two circles AFB, NQD given in magnitude and position, and also the given point A in the line joining their centres; it is required to draw a line through the given point A meeting the circle AFB in F, and the circle NQD in Q, so that the part FQ intercepted between the circumferences may be of a given length. See Fig XXIII. XXIV. &c. to XXXIV.

ANALYSIS. Suppose FQ the given line which passes through A meeting the circle DQN in P and Q, and the circle AFB in F; join AF, and take DR equal, and in a contrary direction to NB, and draw RS parallel to BF, and parallel to RS draw mn through the centre of the circle DPN, meeting QP in n , and bisecting it, because AFB is a right angle; also take the point H so that $AR : AB :: AD : AH$, join DP and draw HK parallel to it, meeting AQ in K; then because $NB = DR$ and $Nm = mD$, $Bm = mR$, and consequently as BF mn and RS are parallel, Fn is equal to nS but $Qn = nP$, therefore $FQ = SP$; now, by similar triangles, $AR : AB :: AS : AF$ and $AD : AH :: AP : AK$, but $AR : AB :: AD : AH$; therefore $AR : AB :: SP : FK$, but SP is equal to QF, therefore $AR : AB :: QF : FK$; but QF is given by hypothesis, therefore FK, and consequently QK, are also given: also the angle $AHK = ADP$, and $ADP = AQN$ by Euclid, Pr. XXXII. Book vi. therefore the angles AHK and AQN are equal, and consequently $AK \cdot AQ = AH \cdot AN$. Hence this

CONSTRUCTION. Take DR equal, and in a contrary direction to NB, and

and let qf be the given line, also take $AR : AB :: AD : AH$, and let fk be a fourth proportional to AR , AB and qf ; then find the point a so that $qa \cdot ak = NA \cdot AH$, and take $AQ = aq$, then QF will be the line required.

DEMONSTRATION. Join DP , NQ and BF , draw mn and RS parallel to BF , and HK parallel to DP ; then because the angle $AHK = ADP$ and $ADP = AQN$, therefore $AHK = AQN$, and the triangles AHK , AQN are similar; wherefore $AH \cdot AN = AK \cdot AQ$ but $AH \cdot AN = ak \cdot aq$, and $AQ = aq$, therefore $ak = AK$ and $KQ = kq$; but because $BN = DR$ and $Nm = mD$, $Bm = mR$, and $Fn = nS$, but $Qn = Pn$, therefore $FQ = PS$; also because $AR : AB :: AD : AH$, and $AR : AB :: AS : AF$ and $AD : AB :: AP : AK$, therefore $AR : AB :: SP : FK$, but $SP = QF$, therefore $AR : AB :: QF : FK$; but by Construction, $AR : AB :: qf : fk$, wherefore $QF : FK :: qf : fk$, and consequently $QF : QK :: qf : qk$, and $QK = qk$, therefore $QF = qf$.

COROLLARY. Because $AR : AB :: AD : AH$, therefore $AR : AB :: NB : BH$, but $AR : AB :: QF : FK$, therefore $NB : BH :: QF : FK$; hence it also follows that if the circles whose diameters are AB and DN intersect each other, and if $AD : AN :: AH : AG$, then will the circle described on HG pass also through the same point of intersection with the other two circles; for as QF has a constant ratio to FK , and the points Q and F coincide at the point of intersection, K will also fall then in the same point.

DETERMINATION of the Limits,

BECAUSE $AK \cdot AQ = AH \cdot AN$, and the locus of the point Q is a circle, the locus of the point K will also be a circle by Prop. IX of Apollonius; and if AG be a fourth proportional to AD , AN and AH , and on GH a circle be described, its circumference will be the locus of all the points K ; and as QF is in a constant ratio to QK

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it is evident that QF will be the least or greatest when QK is least or greatest; and the circles GKH and NQD being similarly situated with those in the first Problem, the limits may be determined in the same manner; or according to the following method.

FIGURE XXIII. Because $AK \cdot AQ = AH \cdot AN$, therefore when AQ is greatest, QK is also greatest, and when AQ is least, QK is least by the Lemma, and consequently QK which is to QF in the constant ratio of NH to NB ; but AQ is least when it becomes AN , and greatest when it becomes the tangent At ; therefore BN and ft are the least and greatest limits.

FIG. XXIV. Here AQ is least when it becomes At , and greatest when AN , therefore by the same reason as the last ft and BN are the least and greatest limits.

FIG. XXVII. and XXIX. By the same method as above, tf here is least and NB the greatest.

FIG. XXVIII. Here tf is greatest and NB the least for the same reason.

FIG. XXX. Here tf and NB are both the greatest limits.

FIG. XXVI. Here AH is either greater, equal, or less than AN : *First* let them be equal; then as $AN \cdot AH = AK \cdot AQ$, and AQ is less than AN ; QK will be greater than HN , and when AQ is least, or becomes At , QK will be greatest, and also QF which is to QK in a constant ratio; hence HN is least and ft greatest. *Secondly*, let AH be greater than AN ; then because AQ is less than AN , and AK greater than AH , QK will by the Lemma be greater than NH , and therefore the limits are as in the former case. *Thirdly*, let AH be less than AN ; then as AQ decreases, and AK increases, they either become equal before QF touch the circle DQN , or not; if they do not, it follows from the Lemma, that as QA and AK approach towards a state of equality, QK decreases, and consequently

QF

QF cannot be greater than BN, nor less than ft ; but if they arrive at equality before Q comes to t , then to find QF the least, draw the line KQ through A, so that $KA=AQ$, then KQ will be the least possible, and consequently QF, which is in a given ratio thereto; and the other limits are still ft and NB, as before.

FIG. XXV. Let $AD:AN::AH:AG$, and on GH describe a circle which will be the locus of all the points K; now AH is either greater, equal, or less than AN; if AH be greater than AN, draw KAQ so that $KA=AQ$, then because $HA \cdot AN = KA \cdot AQ$ and $KA=AQ$, KQ will be the least line that can be drawn between the peripheries, and consequently FQ will be a minimum also; and the maximum will be when KQ is a tangent to NQP; for as K must fall between H and the point where ft touches the circle HKG (otherwise the point Q would be beyond the point of contact t , which would bring it under the case of Fig. XXVI.); therefore as AK is least when AK touches the circle HKG, or AQ touches NQD, BN and ft are the limits of QF: If AH be equal to AN, then HN will be less than KQ, and the maximum will also be when KAQ touches the circle NQP; for as KA decreases, KA and AQ differ the more from equality, and of consequence KQ increases, and also QF: If AH be less than AN, then because KA is less than AH, and AQ greater than AN, KA and AQ differ more from equality than HA and AN; therefore KQ is greater than HN, and increases till it becomes the tangent, and the limits of QF are ft and BN.

FIG. XXXIII. Because $AR:AB::NB:BH$, and $DB=NR$, therefore if AD be less than AN, AR is greater than AB, and consequently NB greater than BH, therefore the point H falls within the circle; and as $AK \cdot AQ = AH \cdot AN$ and AQ is less
F than

than AN, KQ is less than NH, and as AQ decreases till it arrives at T, so QK decreases till the point F comes to A; therefore QF cannot be less than AT, nor greater than BN. If AN be less than AD, then NB will be less than Bh, and h falls without the circle; and because $AN \cdot Ah = AQ \cdot Ak$ and AQ increases, therefore Qk decreases and consequently QF; therefore in this case also, AT and BN are the limits of QF. If AD be equal to AN, then $AR = AB$, and $NB = BH$, therefore H falls upon N, and as AQ is constant, and AF decreases till F comes to A, AT and BN are still the limits of QF.

FIG. XXXII. Because $AR : AB :: NB : BH$, and DR is equal to BN, if AD be greater than AN, AR is greater than AB, and if less, less, and NB greater, or less than BH respectively, and H falls accordingly either without or within the circle NQD; and as the rectangle NAH is equal to KAQ, if H fall without the circle; as AQ increases QK (and consequently QF, which is a given ratio to it) decreases till AK and AQ become equal and the circles intersect, and QK and QF vanish; and afterwards QK increases again till AQ becomes AT, which is evidently its Limit, for if it was to move farther it would become the case of Fig. XXXIII.

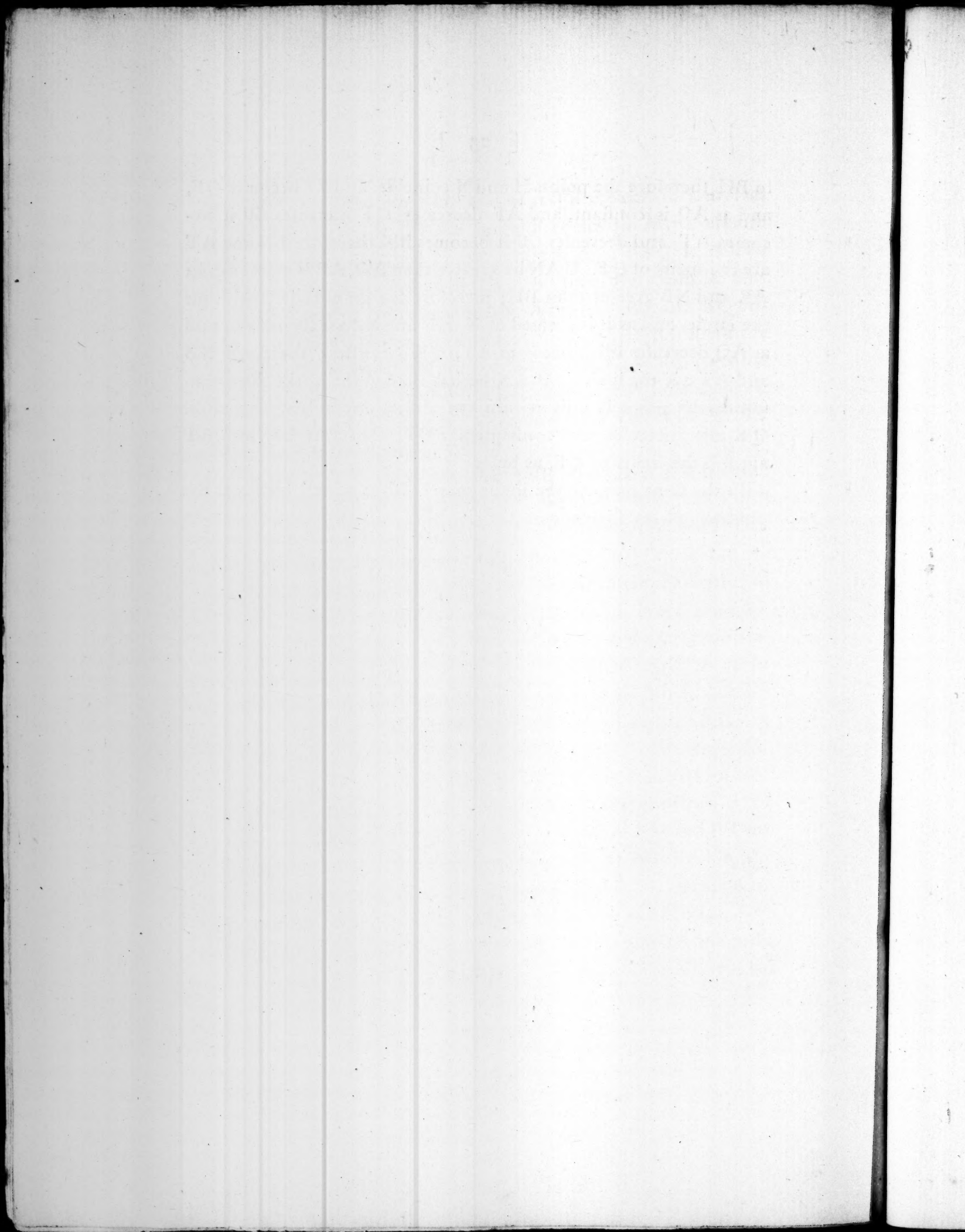
FIG. XXXIV. If AN be less than AD, AR is less than AB, because $AR : AB :: NB : BH$, therefore NB is less than BH and the point H falls without the circle, and QAK being constant, while AQ increases towards T, QK decreases; therefore NB and AT are the limits of QF: if AN be greater than AD, H falls within the circle for the same reason as above, and AQ and QK decrease together and consequently QF; and the limits are as before.

FIG. XXXI. Here DA and AN are either equal or unequal; if equal, because $AR : AB :: NB : BH$ and AR equal AB, NB is equal

to BH, therefore the points H and N coincide, and KF becomes QF, and as AQ is constant, and AF decreases, QF increases till it becomes AT, and decreases till it becomes BN, therefore BN and AT are the limits of QF. If AN be greater than AD, AB is greater than AR, and NB greater than BH; therefore the point H falls without the circle, and as it is greatest at N, HN and NB are then least; and as AQ decreases till it becomes AT, QK increases; therefore NB and AT are the limits. If AN be less than AD, by the same reasoning the point H falls within the circle, and as AQ increases, QK also increases and consequently QF, therefore BN and AT are still the limits of QF, as before.

In the same manner the limits may be determined however the positions of the figures may vary.

T H E E N D.



E S S A Y II.

ON THE DOCTRINE OF PROJECTILES.

P R E F A C E.

THE following Constructions are not only of service in resolving the common questions of Gunnery on the parabolic hypothesis, but are likewise applicable to many other problems where the ellipse and hyperbola are similarly concerned; for as in the parabola, lines drawn from a point in the curve to the focus and directrix are equal, so in the other conic sections they are in a constant ratio to each other; from whence the method of Construction here given may be easily applied to the other cases.

TORICELLI seems first to have shewn that the greatest range on a horizontal plane would be made at an elevation of 45 degrees, if the air had no resistance; Dr. Halley afterwards rendered that discovery much more general, and proved that the greatest range, on any inclined plane, was made when the direction of the piece bisected the angle made by the plane and the zenith; yet in both cases it was supposed that the piece itself was in the plane, which seldom happens. But to find the direction to have the greatest range when the piece itself is above or below the given plane, is a much more difficult problem; and as its solution has never

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yet

yet been publicly given, it is added here to render the subject complete.

As the computations may be readily made from the Constructions, the trigonometrical rules are omitted, and the more so as they can be of little use in their present form, without corrections on account of the air's resistance: this evidently follows from the additions made by the learned Mr. Brown to his excellent Translation of Euler's Gunnery. However, lest the Reader should think the book incomplete without them, a small piece is added containing a different method of Construction, and an investigation of the usual rules; which if any one chuses to see exemplified, I would particularly recommend him to that elegant piece on Projectiles in Mr. Simpson's Select Exercises; for as to those productions that bear the names of Waddington, Glennie, &c. they are so contemptible as to entitle their authors to no other part of the public indulgence but "*the running banquet of two beadles.*"

It being often necessary, in the practice of Gunnery, and on many other occasions *to extract the cube root*, I have, notwithstanding it is rather out of its place, given here the following rule for performing that operation, which will be found much easier to be remembered than the common method, and far more exact and expeditious.

RULE. *Twice the nearest cube, added to the given number, is to the difference between the given number and the nearest cube, as the root of the nearest cube, to the correction; which must be ADDED to the root of the nearest cube, if the nearest cube is less than the given number, otherwise SUBTRACTED.* This rule is deduced from article XIV. of my Diary for 1778.

E S S A Y, &c.

PROPOSITION I.

THE space descended by a falling body, near the earth's surface, is as the square of the time.

FOR the space descended is as the time and the velocity, but the velocity being uniformly accelerated, is as the time; therefore the space is as the square of the time.

If d represent the distance, t the time, and v the velocity; then as the space descended in the first second is found by experiment to be nearly 16 Feet, we shall have this equation, $d = 16t^2$: also, as the body in falling any distance, acquires a velocity capable of carrying it uniformly over twice that distance in the same time, or through a space of 32 feet; and as the velocity is as the time, therefore 1 second is to 32 feet as the time to the velocity, and consequently $v = 32t$; from which equations we have the following expressions for each quantity, viz. $d = \frac{1}{16}v^2 = 16t^2$; $v = 8\sqrt{d} = 32t$; and $t = \frac{1}{8}\sqrt{d} = \frac{1}{32}v$.

COROLLARY. Hence the spaces described in equal successive portions of time, are as the numbers 1, 3, 5, 7, &c.

PROPOSITION II.

THE curve described by a projectile in a non-resisting medium is a parabola.

LET the body be projected from A in the direction AC (See Fig. XXXV.), and describe the curve AD, then the motion of the body
may

may be resolved into two motions; one of which, in direction AC, is uniform, and therefore is as the time; the other, in direction CD, arises from the accelerating force of gravity, and therefore is as the square of the time, by Prop. I.; wherefore CD is as the square of AC, and consequently the curve is a parabola.

PROPOSITION III.

THE velocity of a Projectile, in any point of the curve, is the same as the body would acquire by falling through a space equal to the fourth part of the parameter belonging to that point.

As the body describes AD by the compound forces of gravity and projection, in the same time as it would describe CD and AC by these forces separately; and as the velocity acquired by falling through CD would carry the body uniformly through twice CD in the same time, therefore the velocities being as the spaces, twice CD is to AC as the velocity at D to the velocity at A; now, in order to find the ratio of CD to AC, when the velocity acquired by falling through CD is the same as the velocity at A, we have only to suppose the two first terms of the proportion equal, *viz.* AC, and twice CD, and we have AC^2 equal to $4 CD^2$; but AC^2 is equal to the rectangle of CD and the parameter, therefore the parameter is equal to 4 times CD, or CD one fourth of the parameter, and hence the velocity of projection at any point A, is the same as the body would acquire by falling through one fourth of the parameter belonging to that point.

COROLLARY I. Hence the velocity at any point A is equal to that acquired by falling from the directrix of the parabola to that point; for the distance between the point and the directrix is equal to the distance from the point to the focus, which is known to be
one

one fourth of the parameter: this distance, when A is the point of projection, is usually called the impetus.

COROLLARY II. Hence also, the range on a horizontal plane, at an elevation of 45 degrees, is twice the impetus; for in that case, the focus is in the point bisecting the horizontal range, and the distance from thence to the point of projection, is equal to the distance of the point of projection from the directrix.

PROPOSITION IV.

GIVEN the direction and impetus of the Projectile, to find where the curve described thereby meets a plane passing through the point of projection. See *Fig. XXXVI. and XXXVII.*

LET AP be the plane, AB the direction of projection, and AC the impetus or space through which the body must fall to acquire the velocity of projection; continue AP to meet CD (drawn perpendicular to AC) in D, make the angle BAF=BAC and AF=AC; join DF, and from A, with the distance AC, describe a circle cutting DF in *n*; join An and draw FP parallel thereto, then P is the point required.

FOR draw PE perpendicular to CE, then because An is parallel to PF, DA : DP :: An : PF :: AC : PE, but An is equal to AC, therefore PF is equal to PE, therefore the point P is in the parabola.

PROPOSITION V.

GIVEN the direction and velocity of projection, to find the point where the Projectile meets a plane given in position. See *Fig. XXXVIII. XXXIX. and XL.*

LET AC be the impetus, AR the direction, and BP the plane, it is required to find the point P where a body projected in the direction AR meets the plane BP.

H

PRODUCE

PRODUCE PB to meet DC (which is normal to AC) in D; and produce AC to cut the plane BP in B, make the angle $RAF = RAC$, and AF equal to AC, and join DF, and from the centre B, with the distance BC, describe a circle cutting DF in n , join Bn , and parallel thereto draw FP, which cuts the plane in the point P required.

For $AC = AF$ by Construction, also $DB : DP :: Bn : PF :: BC : PE$, but $Bn = BC$, therefore $PF = PE$, and consequently the point P is in the parabola.

PROPOSITION VI.

GIVEN the velocity and direction, to find in what part of a given plane a piece ought to be placed to hit an object in the same plane. See *Fig. XLI. and XLII.*

LET P be the object, PA the plane, PB the direction, and PE perpendicular to the horizon. Make $BPM = BPE$, and PR the impetus; in PM, PE take any two equal lines Pm, Pe ; draw ed perpendicular to PE and join dm ; take $Pf = Pe$ or Pm , and continue it if necessary, and draw RF parallel to PA, cutting Pf in F, then FA drawn parallel to PR gives the point A required.

For draw FE, EM parallel to fe, em , and continue MF to meet PA in D and join DE, and draw AC parallel to PE; then because PF, PE and PM, as well as Pf, Pe, Pm are equal, DM and dm are parallel, therefore $Pf : PF :: Pd : PD :: Pe : PE$, therefore de is $\parallel$ to DE; also $DA : DP :: AF : PM :: AC : PE$, and therefore $AC = AF$, and $FP = PE$, consequently a body projected from A with the impetus AC, and direction bisecting the angle CAF (equal EPM), will strike the point P.

PROPO-

PROPOSITION VII.

GIVEN the velocity of projection; required the direction so that the Projectile may pass through a given point. See *Fig. XLIII. and XLIV.*

LET *A* be the point from whence the projection is to be made, *AC* the impetus, and *P* the given point.

PERPENDICULAR TO *AC* draw *CE*, join *AP*, and draw *PE* at right angles to *CE*, and from the centres *A* and *P*, with the distances *AC* and *PE*, let two circles be described intersecting each other in *F* and *f*, then the arcs *CF*, *Cf* being bisected will give two directions, either of which will answer the purpose.

FOR because $CAB = BAF$, *AB* is the tangent, and *F* the focus, and $FP = PE$, therefore the point *P* is in the parabola. This solution has been given by other writers.

PROPOSITION VIII.

GIVEN the direction and velocity, required to find in what part of a plane a piece must be placed so as to hit a given object above or below the given plane. See *Fig. XLV.*

LET *AB* be the given plane, and *P* the given object, in *AB* assume any point *m*, and find where a Projectile from thence, with the given direction and velocity, would meet a plane *PR* drawn parallel to *AB*, let *n* be that point, then take *mM* equal to *nP*, and *M* will be the point required; or it may be otherwise constructed, in a direct manner similar to that of Prop. VI.

PROPOSITION IX.

GIVEN the direction of projection, to find the velocity so that the body may pass through a given point. See *Fig. XLVI. & XLVII.*

LET *AB* be the direction given, and *P* the given point; draw

Am

Am parallel to the horizon, and AC perpendicular to it, and draw AF making the angle BAF equal to BAC ; from P draw the perpendicular Pm to Am , and with that distance describe a circle, and in AF take Aw equal to its radius, join wP , and draw As parallel to it cutting the circle in s , then Ps being joined will cut AF in F , and $AC=AF$ will be the impetus required.

For draw CE perpendicular to AC , then because As is parallel to Pw , and $Aw=Ps$, AF will be $=Fs$, but $AF=AC=Em$, therefore $Fs=Em$ and $FP=EP$, therefore the point P is in the parabola.

PROPOSITION X.

GIVEN the projectile velocity, required the direction so that the range on a plane, passing through the point of projection may be the greatest possible. See *Fig. XLVIII.*

Let AC be the impetus as before, and AP the plane; then let AB be drawn to bisect the angle CAP , and AB will be the direction required.

For let Af be any other direction, and Ap the range answering to it (in the same manner as AP was the range answering to AB), and take $fAd=CAf$, and draw Dp parallel to Ab , which gives p ; wherefore $Db:Dd::DA:Dp$, also $Dn:DF::DA:DP$, but the ratio of Db to Dd is greater than the ratio of Dn to DF , consequently the ratio of DA to Dp is greater than the ratio of DA to DP , and therefore DP is greater than Dp , and consequently DP is the greatest range sought.

PROPOSITION XI.

HAVING given the Projectile velocity, it is required to find the direction, so that the range on a plane given in position may be the greatest possible.

DRAW

DRAW DCE perpendicular to AC, and let AC the impetus meet BP the given plane in B; from the centres B and A with the radii CA and CB describe two circles, and from their centres draw AS and RB parallel; join RS meeting AB in x , and draw Dx cutting the circles ACS and BCR in F and n , and also join AF meeting the plane in P, and bisect the angle CAF by the line Ab; then Ab is the required direction, and P the point where the Projectile meets the plane.

FOR, assume any other direction than Ab, and suppose it to bisect the angle CA r , and let Dr meet the circle BCR in m , and the circle PE z in t , join B m , and parallel to it draw rp meeting BP in p ; then because BR:AS::B x :Ax::B n :AF, therefore B n is parallel to AFP; and because B n and BC are equal, PF is equal to PE and CA to AF, therefore the points A and P are in a parabola whose focus is F, and because AFP is a right line, a circle described from the centre P, and distance PE touches the circle ACS; also because the centres of the circles RC k and F z E are in the line BP, and DCE is their common tangent, D k :D z ::D n :DF (by Prop. V. Apollonius de Locis planis), but D n :DF::D m :Dt::DB:DP; but the ratio of D m to Dt, is less than the ratio of D m to Dr, or of DB to D p , therefore the ratio of DB to DP is less than the ratio of DB to D p , and therefore D p is less than DP, and consequently DP is the greatest range possible; for p is also a point in the parabola, to which the line bisecting CA r is a tangent, because rp and B m are parallel, and B m equal to BC, therefore pr is equal to pe .

S C H O L I U M.

AS it may be necessary, in some of the foregoing Constructions, to draw a line from a given point through the intersection of two lines given in position, when that intersection is not actually assigned, or falls without the limits of the paper, &c. I would recommend for that purpose the method inserted in the first edition of Brook Taylor's Perspective, a demonstration of which may be seen in p. 38 of my Diary for 1778; but as that operation is not often requisite in some of the simplest problems, especially when the methods of Construction of Mr. Cotes and Dr. Pemberton are employed, I have here subjoined a small piece containing the substance of what has been given by Keill, Gray, Starratt, &c. together with the trigonometrical investigation of the greatest part of Mr. Simpson's Rules, done in a different and simpler method. See *Fig. II. and LII.*

LET AC be a plane given in position, AB the given direction in which a ball is fired, with the velocity acquired by falling through the given space rA ; and let Am be perpendicular to the horizon, and $=4Ar$, and a circle be described through the points A, m , touching AC in A, and meeting AB in B; then if BC be parallel to mA , the ball will meet the plane AC in the point C: here C is supposed to be the object.

FOR by Prop. III. rA is one fourth of the parameter of the parabola described by the ball, therefore Am is the parameter, and because the angle $BAC = BmA$, and $ABC = BAm$, therefore $ABm = ACB$, and the triangles are equiangular, consequently $Am \cdot BC = AB^2$; therefore the point C is in the parabola.

Also the time of descent through mA , the time of describing
 Am

Am uniformly with the velocity acquired by falling through rA , and the times of ascent from A to r and descent from r to A , taken together, are equal.

For suppose two bodies projected directly upwards from A , with a velocity equal to that acquired by falling through rA , and that gravity acts on the first, while the second moves uniformly; then by Prop. I. it is evident that while the first body ascends to r , the other will have moved uniformly twice that distance, or through AX ; and while the first body descends again from r to A , the second will move through Xm , wherefore the times of ascending and descending through Ar and rA , taken together, are equal to the time of equable ascent or descent through Am ; and if ms be one fourth of Am , a body descending from rest at m acted on by gravity, will describe ms while the other ascends through Ar , and the remainder sA while the other descends from r to A , by Cor. to Prop. I., because ms is to sA as 1 to 3.

Also the time of flight in perpendicular projections is to the time of flight in oblique, with the same velocity, as Am to AB .

For the time of describing AB uniformly with the velocity of projection, is equal to the time the ball takes to describe the curve which passes through C , since the velocity, in direction AB (which may in both cases be represented by AB), may be resolved into the two represented by AC and CB , and in these directions; but as AC is the same in both, the ball arrives at C , when acted on by gravity, in the same instant that it would arrive at B with the uniform velocity of projection if gravity did not act; and the time of describing Am is to the time of describing AB as the lines themselves when the velocity is the same.

FROM the above principles the solutions of the following problems may be deduced.

I. When

I. When the direction, the impetus and position of the object are given, to find the time. Let $Am = 4$ times the impetus, AB the direction, and C the object, and let Am and CB be perpendicular to the horizon; then describe a circle through the points A and m touching AC in A , and meeting CB in B , then Am is to AB as the time of flight in the perpendicular projection to that in the direction AB : and because $s \cdot mAC = s \cdot mAR = s \cdot mBA$, and $s \cdot BAC = s \cdot BmA$, and the time of describing Am is $\frac{1}{2}\sqrt{Ar}$ by Prop. I. therefore $s \cdot mAC : s \cdot BAC :: Am : AB :: \frac{1}{2}\sqrt{Ar} : \text{time of flight}$, which gives this rule: *As the sine of the zenith distance of the object, is to the sine of the elevation of the piece above the object, so is half the square root of the number of feet in the impetus to the time required.*

II. When the position of the object and direction of the piece are given, to find the impetus. Draw CB perpendicular to the horizon meeting the direction AB in B , and on AB describe a segment of a circle containing the given angle BAC meeting Am (parallel to BC) in m , then Am is 4 times the impetus; and because $s \cdot ABD$ ($s \cdot mAB$): radius :: $AD : AB$, and $s \cdot BAC$ (BmA): $s \cdot ACD$ ($s \cdot ACB = s \cdot ABm$) :: $AB : Am$, therefore $s \cdot mAB \times s \cdot BAC : \text{rad.} \times s \cdot ACD :: AD : Am$, which gives this trigonometrical rule: *As the rectangle of the sine of the zenith distance of the piece, and the sine of its elevation above the object is to the rectangle of the radius and the sine of the zenith distance of the object, so is the horizontal distance of the object to four times the impetus.*

III. The inclination of the plane, the direction and impetus being given, to find the range on that plane. Let AB be the direction, and A 4 times the impetus, and make the angle AmB equal to BAC , then BC drawn parallel to Am , meeting the plane in C , gives AC the range required. And because $s \cdot mAC$ ($s \cdot mQA$): $s \cdot BAC$ ($s \cdot BQA$) :: $Am : AB$, and $s \cdot mAC$ ($s \cdot ACD = s \cdot ACB$): $s \cdot mAB$ ($s \cdot ABC$): $AB : AC$; therefore $s \cdot mAC : s \cdot BAC \times s \cdot mAB :: Am : AC$, which gives
this

this practical rule: *As the square of the sine of the plane's zenith distance is to the rectangle of the sines of the elevation above the plane, and zenith distance of the piece, so is four times the impetus, to the range required.* And by inverting the terms, the impetus may be found from this proportion, when the range, direction and position of the plane are given.

HENCE also the greatest range on a horizontal plane is twice the impetus; for when AC is horizontal, the centre P falls in X, and Xb or AC becomes AX; and hence, when Ac the greatest range on the inclined plane is given, AX, or the greatest range on the plane of the horizon may be found by finding cd, which is a fourth proportional to radius, the sine of the plane's inclination cAd, and Ac the range given; then Ac plus or minus cd according as the object is elevated or depressed gives bd=AX, the greatest horizontal range required: also when Ac the range on the given plane AC is the greatest, and bc=Ac touches the circle in b, the angle bAc=Abc=bAm, therefore the range is greatest when the direction of the piece bisects the angle between the plane and the zenith: again, when the greatest horizontal range is known, the greatest range on a given plane may be found; for the first being =AX or bd, and the angle mAb or $\frac{1}{2}mAc$, and its complement bAd, being given, and also cAd; therefore $\tan. bAd : \tan. cAd :: bd : dc$, and the sum or difference of bd and dc is =bc=Ac the range sought.

IV. *The impetus and position of the object being given, to find the direction so as to hit the object.* Take Am as before, and draw a circle through the points A and m, touching the given line AC in the point A, and through the object C draw CB parallel to Am, meeting the circle in B; then AB is the requisite direction. Again,

K

because

because the angle $XAP = DAC$, $AD : CD :: AX : XP$, and XP added to, or subtracted from, AD or XZ , according as the object is elevated above the horizon, or depressed, gives PZ ; then $AX : PZ :: s \cdot mAC (= s \cdot mAR = s \cdot XPA) : \cos. BPZ (s \cdot PBZ)$, then half $BPZ = bAB$, and $bAB \pm \frac{1}{2}mAC (bAC) = BAC$; which gives this rule for the calculation. *As the radius is to the tangent of the object's elevation or depression, so is twice the impetus to a fourth proportional, which add to, or subtract from, the horizontal distance of the object, according as it is elevated or depressed; then twice the impetus is to the last result, as the sine of the zenith distance of the object, to the cosine of an angle, half of which being added to, or subtracted from half the zenith distance of the object, gives the two zenith distances of the directions by which it may be hit.*

It is evident that, in horizontal projections, Am is the diameter of the circle, X its centre, and BXA twice the angle BAC or BmA , and also AC equal to the sine of BXA , or twice BAC ; hence the ranges on a horizontal plane, with the same impetus, are as the sines of twice the angles of elevation; and as AX is the greatest horizontal range, $AX : AC :: \text{radius} : s \cdot \text{twice } BAC$; that is, the greatest horizontal range is to the range at a given elevation, as radius to the sine of twice the elevation: also, because the time of descent through Am , is to the time of flight as Am to AB , and $Am : AB :: \sqrt{Am} : \sqrt{BC} :: \frac{1}{4}\sqrt{Am} : \frac{1}{4}\sqrt{BC}$, therefore as $\frac{1}{4}\sqrt{Am}$ is the time of descent through Am , $\frac{1}{4}\sqrt{BC}$ is the time of flight; and it is evident that in horizontal cases AC is to CB , as radius to the tangent of the angle of elevation.

HENCE, by varying the terms of those proportions, we have rules for solving the following problems in horizontal-projections.

I. HAVING the distance of the object and the elevation of the piece, to determine the time of flight.

RULE.

RULE. *As radius is to the tangent of the elevation, so is the given distance of the object in feet to a number, one fourth of whose square root is the number of seconds required.*

II. **HAVING** given the greatest range on a horizontal plane, to find the range at any proposed elevation.

RULE. *As radius is to the sine of twice the given elevation, so is the given range to the range required.*

III. **HAVING** the greatest range given, it is required to find the elevation to strike an object at a given horizontal distance.

RULE. *As the greatest range is to the given distance, so is radius to the sine of twice the angle of elevation.*

IV. **THE** elevation and distance of the object being given, required the impetus so as to strike the object.

RULE. *As the sine of twice the elevation is to radius, so is the given distance of the object to twice the impetus.*

V. **THE** range at one known elevation being given, to find the range at any other known elevation.

RULE. *As the sine of twice the first elevation is to the sine of twice the second, so is the range at the first elevation to that at the second.*

THE END.

E R R A T A.

In page 17, line 11, and page 20, line 3, for *first* read *third*.

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MVSEVM
BRITAN
NICVM

Plate 1.

